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Ex: Every plane (in $\mathbb{R}^3$) can be parametrized by $\vec{s}(u,v) = u\vec{a} + v\vec{b} + \vec{c}$ for suitable $\vec{a}, \vec{b}, \vec{c}$
for $D = \mathbb{R}^2$

Idea: π is just determined by points (u,v) in $\mathbb{R}^2$ via $\vec{a}, \vec{b}, \vec{c}$ and the equation above

Ex: compute a parametrization for the paraboloid $z = x^2 + 2y^2$

NB: There are many ways to parametrize this surface.

Sol ①: $\vec{s}(x,y) = \langle x, y, x^2 + 2y^2 \rangle$ $D = \mathbb{R}^2$

Sol ②: $\vec{s}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), (r\cos(\theta))^2 + 2(r\sin\theta)^2 \rangle$
$= \langle r\cos(\theta), r\sin(\theta)\ r^2(1+\sin^2(\theta)) \rangle$
$D = [0, \infty) \times [0, 2\pi]$

Sol ③: $\vec{s}(r,\theta) = \langle \sqrt{2}\, r\cos(\theta), r\sin(\theta), 2r^2 \rangle$

Ex: Let $f(t)$ be a single-variable function. The surface defined by revolving f about the x-axis is parametrized by
$\vec{s}(x,\theta) = \langle x, f(x)\cos(\theta), f(x)\sin(\theta) \rangle$
↳ Let $f(x) = x^3$
This surface has parametrization $\rightarrow$

$\vec{s}(x,\theta) = \langle x, x^3\cos(\theta), x^3\sin(\theta) \rangle$

11/22/21 Surfaces and Calculus: A surface in $\mathbb{R}^3$ has the form $\vec{s}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ on some domain $D \subseteq \mathbb{R}^2$

Ex: The torus w/ major radius $\alpha > 0$ and minor radius β (w/ $\alpha > \beta > 0$) is the surface $\vec{s}(u,v) = \langle (\alpha + \beta\cos(u))\cos(v), (\alpha + \beta\cos(u))\sin(v) \rangle$

$\rightarrow$

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I. Tangent Planes

The tangent plane to surface $\vec{s}(u,v)$ at point $(a,b) \in D$ has normal vector $\vec{n}(a,b) = \vec{s}_u(a,b) \times \vec{s}_v(a,b)$

NB: $\vec{S}_u = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \rangle$, can also be written $\frac{\partial \vec{s}}{\partial u}$

Ex: consider the torus $\vec{s}(u,v)$ w/ major radius 10 and minor radius 5. What is the tangent plane to $\vec{s}(u,v)$ at $\vec{s}(\frac{\pi}{4}, \frac{3\pi}{4})$

Sol: $\vec{s}(u,v) = \langle (10+5\cos(u))\cos(v), (10+5\cos(u))\sin(v), 5\sin(u) \rangle$

$\vec{S}_u = \langle -5\sin(u)\cos(v), -5\sin(u)\sin(v), 5\cos(u) \rangle$

$\vec{S}_v = \langle -(10+5\cos(u))\sin(v), (10+5\cos(u))\cos(v), 0 \rangle$

normal vector

$\vec{n}(u,v) = \vec{S}_u(u,v) \times \vec{S}_v(u,v)$ ← (NB: in principle, we could have computed $\vec{s}_u(\frac{\pi}{4}, \frac{3\pi}{4}) \times \vec{s}_v(\frac{\pi}{4}, \frac{3\pi}{4})$)

$$\begin{vmatrix} i & j & k \\ -5\sin u\cos v & -5\sin u\sin v & 5\cos u \\ -(10+\cos u)\sin v) & (10+5\cos u)\sin v & 0 \end{vmatrix}$$

$= (0 - 5\cos u(10+5\cos u)\cos v)i - (0 - 5\cos(u)(-(10+5\cos u)\sin v))j$

$+ (-5\sin(u)\cos v(10+5\cos u)\cos v - 5\sin u\sin v(10+5\cos u)(\sin v))k$

$= -5\cos(u)\cos(v)(10+5\cos(u))i$

$-5\cos(u)\sin v(10+5\cos u)j$

$-5\sin u(10+5\cos u)(\cos^2 v+\sin^2 v)k$

$= -5(10+5\cos u)\langle \cos u\cos v, \cos u\sin v, \sin u \rangle$

↑ at every $(u,v) \in dom(\vec{s})$, this is the normal vector at $\vec{s}(u,v)$

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Now at the point:

$\vec{S}(\pi/4, 3\pi/4) = \langle (10+5\cos(\frac{\pi}{4}))\cos(\frac{3\pi}{4}), (10+5\cos(\frac{\pi}{4}))\sin(\frac{3\pi}{4}), \sin(\frac{\pi}{4}) \rangle$

$= \langle (10+\frac{5}{\sqrt{2}})\cdot -\frac{1}{\sqrt{2}}, (10+\frac{5}{\sqrt{2}})\cdot\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$= \langle -\frac{10}{\sqrt{2}} - \frac{5}{2}, \frac{10}{\sqrt{2}} + \frac{5}{2}, \frac{1}{\sqrt{2}} \rangle$

$\vec{n}(\frac{\pi}{4}, \frac{3\pi}{4}) = -5(10+5\cos(\frac{\pi}{4}))\langle \cos(\frac{\pi}{4})\cos(\frac{3\pi}{4}), \cos(\frac{\pi}{4})\sin(\frac{3\pi}{4}), \sin(\frac{3\pi}{4}) \rangle$

$= -5(10+\frac{5}{\sqrt{2}})\langle \frac{1}{\sqrt{2}}\cdot -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$= -25(2+\frac{1}{\sqrt{2}})\langle -\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}} \rangle$

The tangent plane at this point is given by

$\vec{n}\cdot(\vec{x}-\vec{p}) = 0$

i.e. $\vec{n}(\frac{\pi}{4}, \frac{3\pi}{4})\cdot(\vec{x} - \vec{S}(\frac{\pi}{4}, \frac{3\pi}{4})) = 0$

i.e. $-25(2+\frac{1}{\sqrt{2}})\langle -\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}} \rangle \cdot \langle x + \frac{10}{\sqrt{2}} + \frac{5}{\sqrt{2}}, y - \frac{10}{\sqrt{2}} - \frac{5}{\sqrt{2}}, z - \frac{1}{\sqrt{2}} \rangle = 0$

i.e. $-\frac{1}{2}(x + \frac{10}{\sqrt{2}} + \frac{5}{\sqrt{2}}) + \frac{1}{2}(y - \frac{10}{\sqrt{2}} - \frac{5}{\sqrt{2}}) + \frac{1}{\sqrt{2}}(z - \frac{1}{\sqrt{2}}) = 0$

II. Surface area

The surface area of a surface $\vec{S}(u,v)$ parametrized on domain D is $A = \iint_D |\vec{S}_u \times \vec{S}_v|$

← area of a parallelogram

Q: where is this coming from?

A: Piecewise approximation of surface S via parallelograms. Limiting these approximations yeilds that formula.

NB: for this to work, we assume that $\vec{S}(u,v)$ traverses the surface once on D. (similar to arclength needs curve to be traversed once)

Ex: compute the surface area of the torus w/ major radius 10 and minor radius 5

Sol: we already computed $\vec{n}(u,v) = \vec{S}_u(u,v)\times\vec{S}_v(u,v) =$

$-5(10+5\cos(u))\langle \cos u\cos v, \cos u\sin v, \sin v\rangle$

→

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$|S_u(u,v) \times S_v(u,v)| = |-5(10+5\cos u)| \sqrt{\cos^2 u \cos^2 v + \cos^2 u \sin^2 v + \sin^2 u}$

$= 25|2+\cos u| \sqrt{\cos^2(u)(\cos^2(v)+\cos^2(v)) + \sin^2(u)}$

$= 25(2+\cos u)$

$\text{Area}(S) = \iint_{D(S)} |\vec{S}_u \times \vec{S}_v| \, dA = \int_{u=0}^{2\pi} \int_{v=0}^{2\pi} 25(2+\cos u) \, dv\, du$

$= 25 \int_{u=0}^{2\pi} (2+\cos u)[v]\Big|_{v=0}^{2\pi} du$

$= 50\pi \left[2u + \sin(u)\right]\Big|_{u=0}^{2\pi}$

$= 50\pi(2(2\pi - 0) + (0-0))$

$= 200\pi^2$

Exersize: Compute the surface area of a general torus of major radius α and minor radius β (should be $4\alpha\beta\pi^2$)

NB: If $f(x,y)$ is a function, the graph is a surface $\vec{S}(x,y) = \langle x, y, f(x,y)\rangle$. The normal vector to this surface is

$\vec{n}(x,y) = \vec{S}_x \times \vec{S}_y = \det \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \langle -f_x, -f_y, 1\rangle$

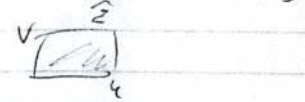
$\therefore \text{Area}(\text{graph}(f)) = \iint_D \sqrt{f_x^2 + f_y^2 + 1} \, dA$

Idea: surface area is an area. So we should (by analogy to previous work) be able to write

$\text{Area}(S) = \iint_S 1 \, dS$

resembles formula $A(R) = \iint_R 1 \, dA$,

to make analogy work, $dS = |\vec{S}_u \times \vec{S}_v| \, dA$



NB: $|\vec{S}_u \times \vec{S}_v|$ is a Jacobian.

→

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III Surface Integrals

The integral of a function $f(x,y,z)$ over a surface S parametrized by $\vec{s}(u,v)$ on domain D is

$$\iint_S f \, dS = \iint_D f(\vec{s}(u,v)) \, |\vec{s}_u \times \vec{s}_v| \, dA$$

NB: $\int_C f \, d\vec{r} = \int_{\text{dom}(c)} f(r(t)) \, |\vec{r}\,'(t)| \, dt$ ← line integral

Each piece is replaced by a 2-dimensional counterpart.